Quasiperiodic Tilings With 12-Fold Rotational Symmetry Made of Squares, Equilateral Triangles, and Rhombi

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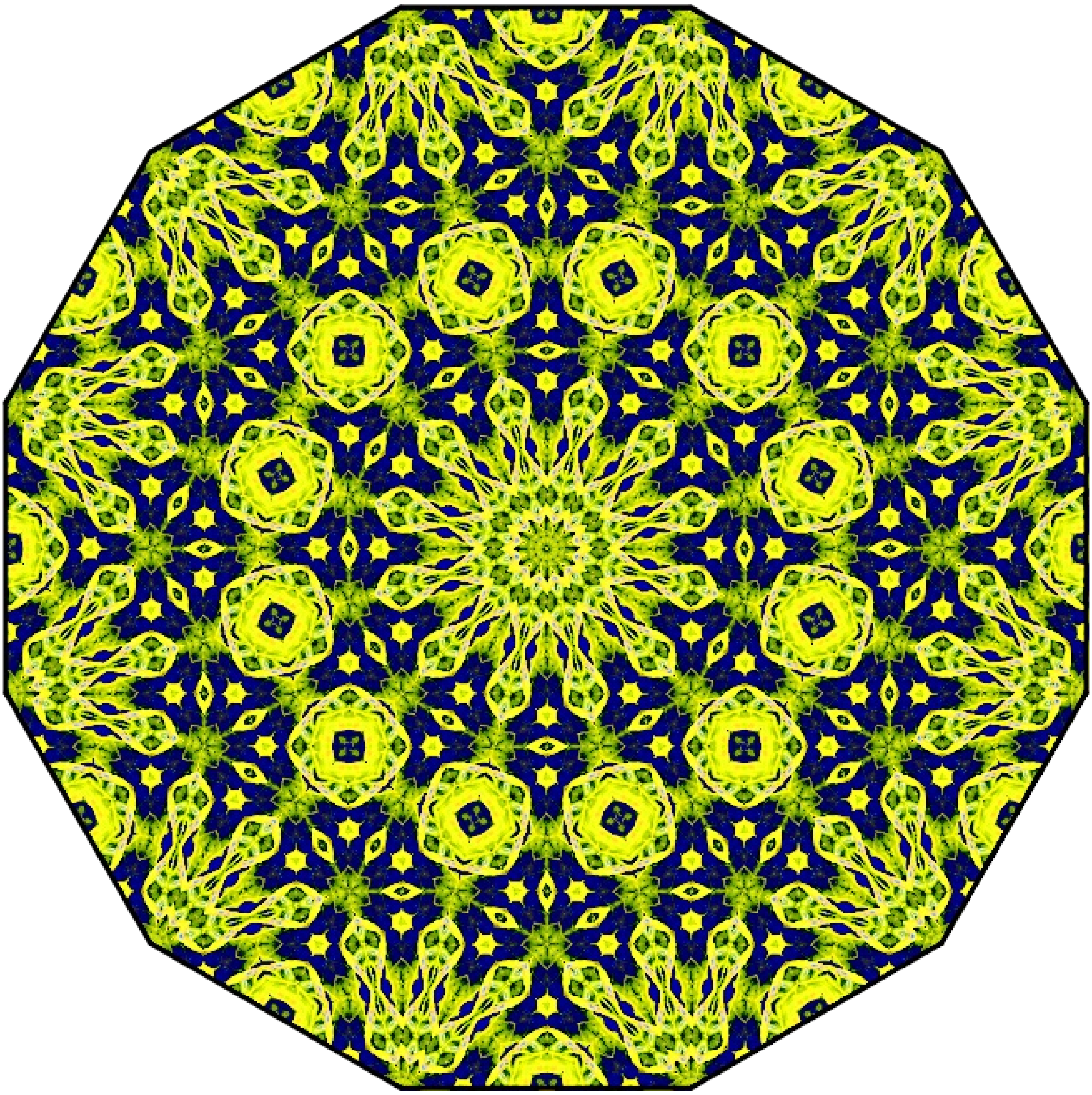


Going back 4000 years ...

About 2000 B.C.: Decoration of a spindle with rosettes of 6 intersecting circles.

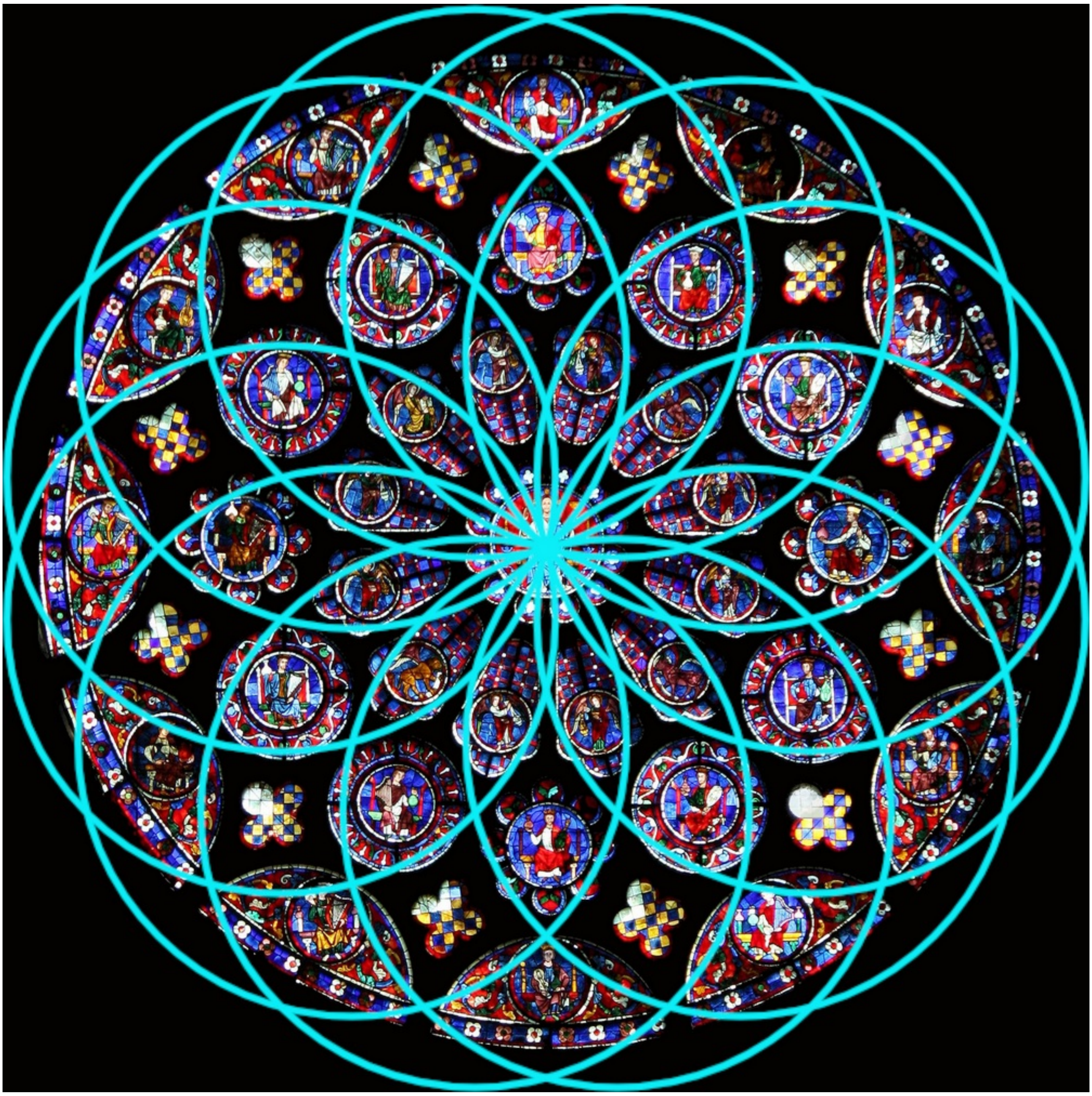
(Britisch Museum, https://www.britishmuseum.org/collection/object/G_1897-0401-1368)

This rosette can be repeated periodically.

With more intersecting circles we can get images of higher symmetry.

About 3000 years later ...

1221-1230 A.C.: The southern rose window of the Notre-Dame Cathedral in Chartres, France, matches a rosette of 12 circles.



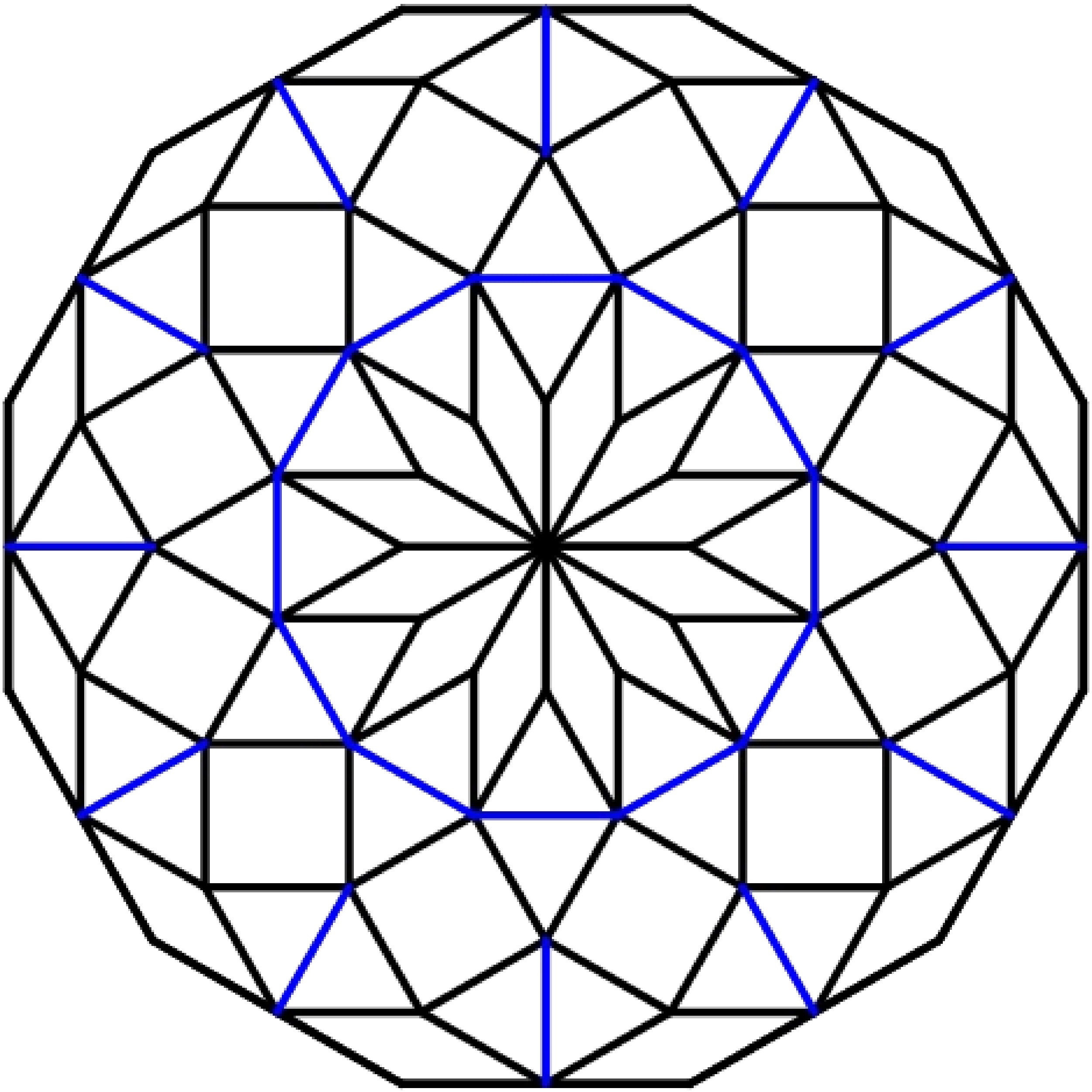
(See https://www.medart.pitt.edu/image/France/Chartres/Chartres-Cathedral/Windows/Transept-windows/122A-South-Rose/Chartres-122SouthRose.HTM)

This rosette has 12-fold rotational symmetry and cannot be repeated periodically.

And now ...

Approximate the circles by dodecagons.

Divide the rhombi of 60 degree angles into two equilateral triangles.
This gives a rosette with squares, equilateral triangles and rhombi of 30 degree acute angle.



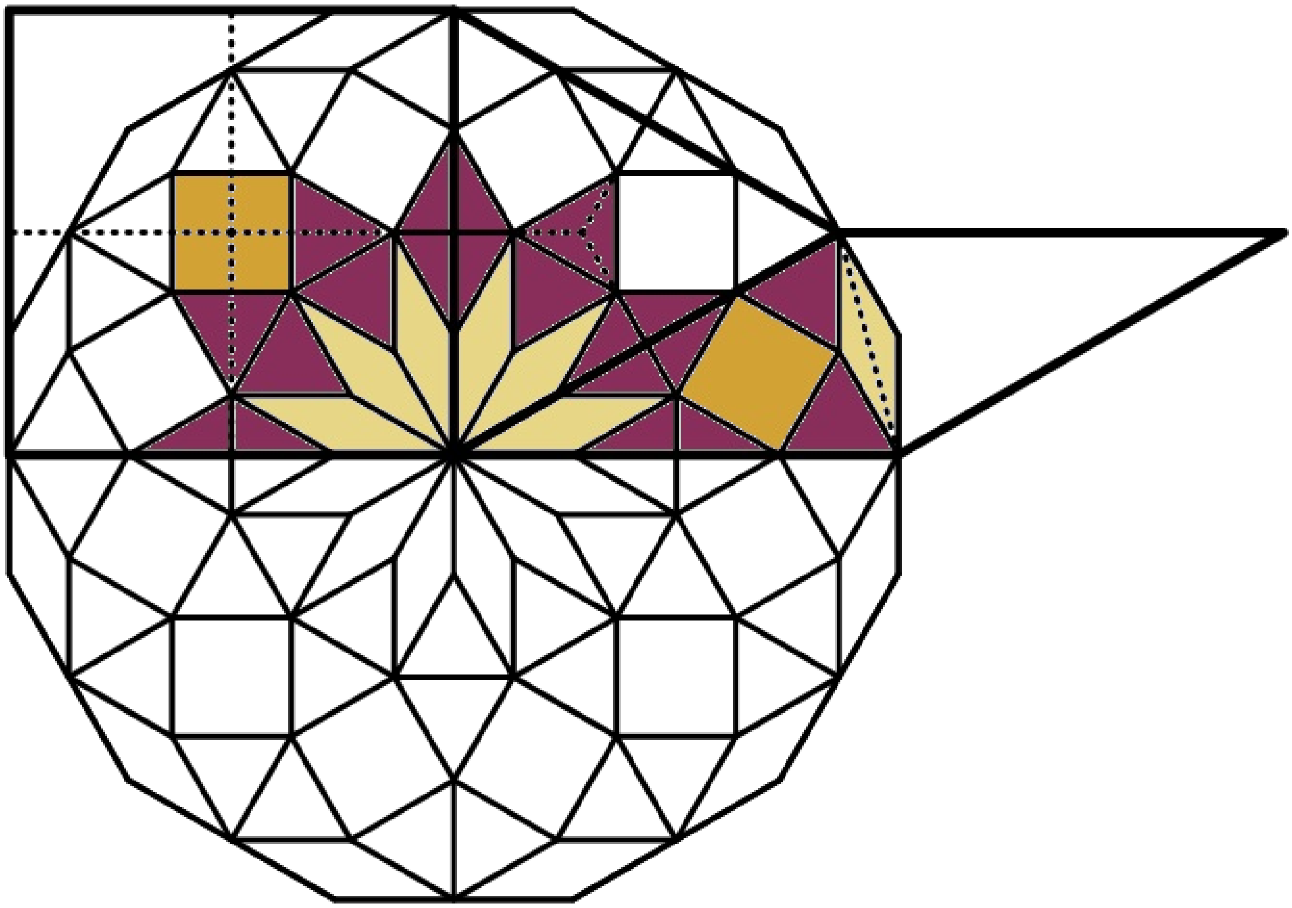
How can we extend this rosette?

Solution: Make a quasiperiodic tiling!

The substitution method is often used for fractals (Koch snowflake, Sierpinsky triangle) or space-filling curves.

We use it to create quasi-periodic tilings.

Large squares, rhombi and equilateral triangles fit the 12-fold rosette if they have a relative length of 2+sqrt(3).



This suggests certain subdivisions of the large polygons into smaller ones.

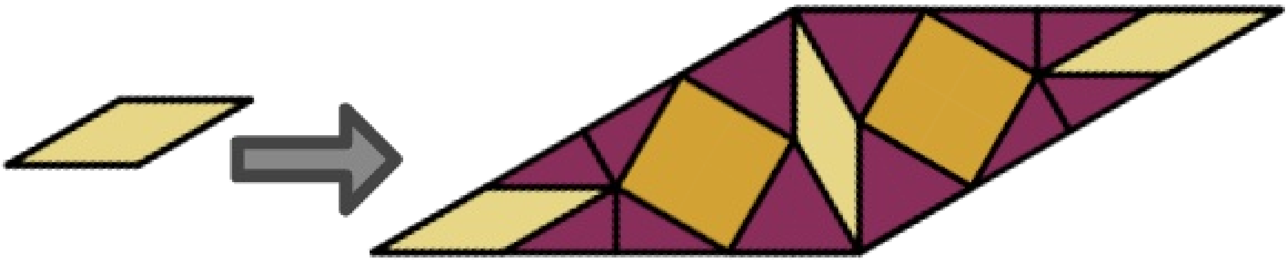
We take only small parts (solid colors) from the rosette and paste them symmetrically on the polygons.

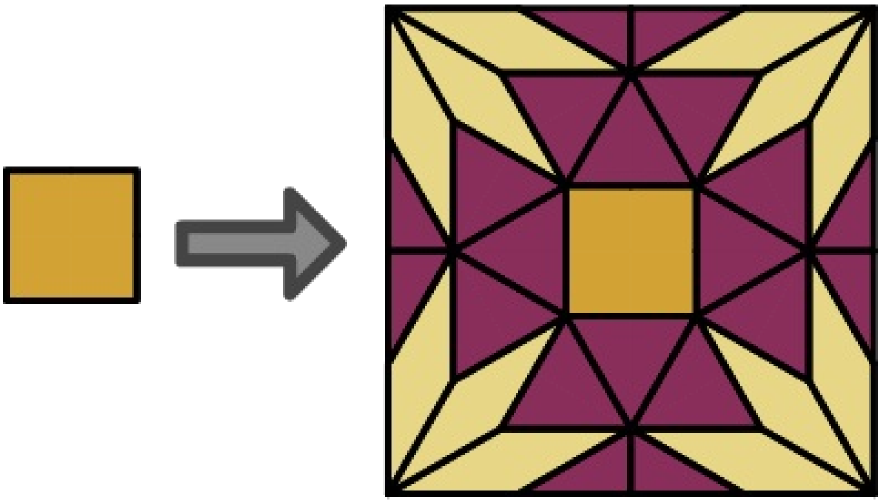
Substitution with inflation ratio 2+sqrt(3)

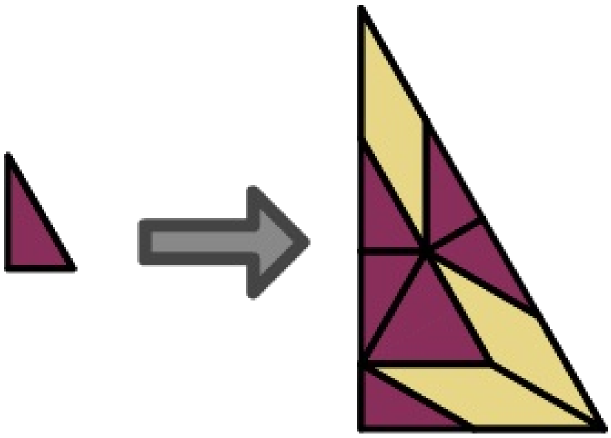
The substitution method has two steps:

1) Increase tile size by the inflation ratio of 2+sqrt(3).

2) Replace the large tile by a symmetric arrangement of small tiles.







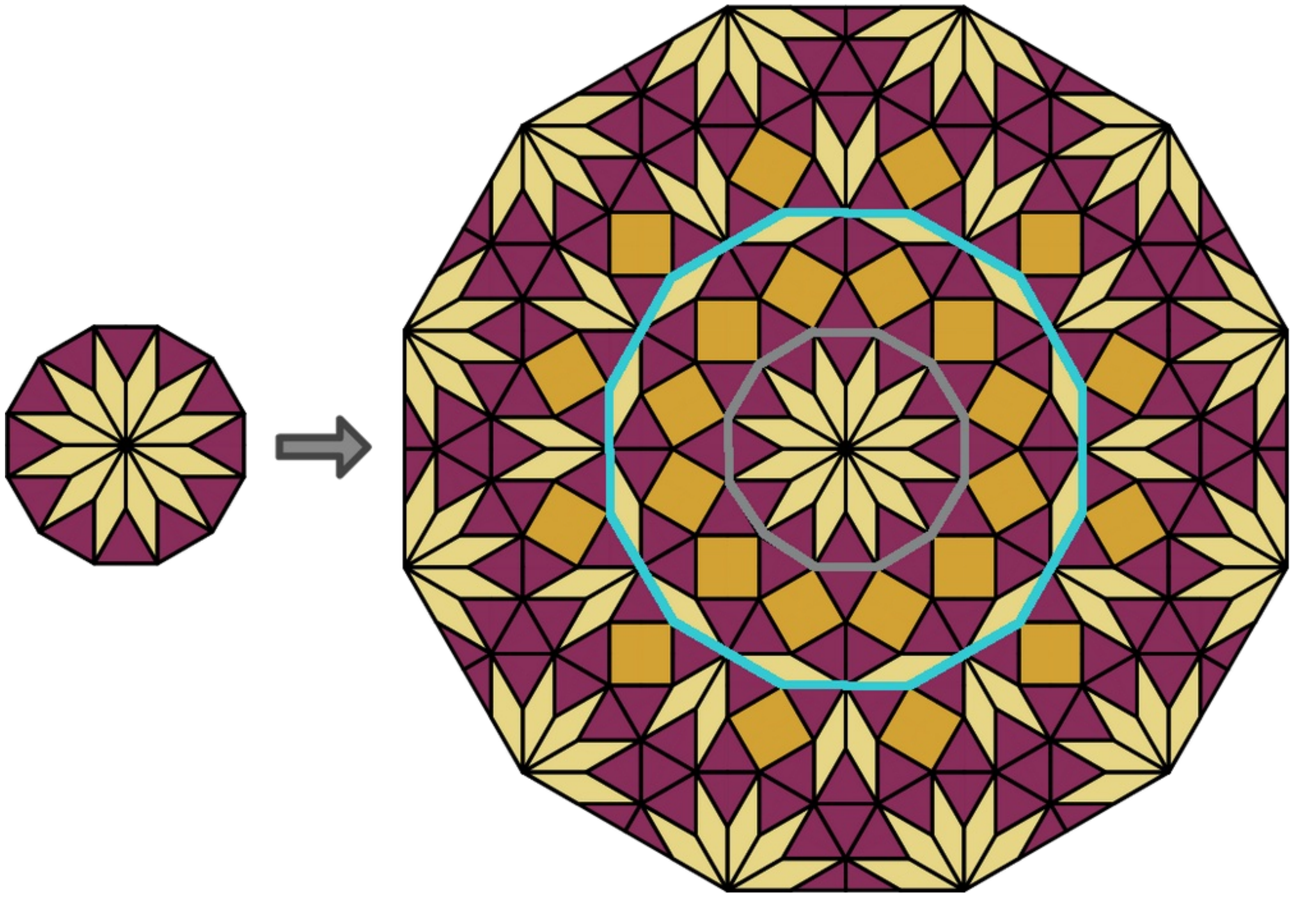
(We use halves of equilateral triangles. This preserves the overall shape of the tiles.)

Doing a substitution step

Using this scheme we can grow an infinite dodecagonal quasiperiodic rosette.

Start with 12 rhombi and 12 triangles.

The first step:



The grey outline marks the initial dodecagon, which gets reproduced.The substitution step essentially adds a layer.

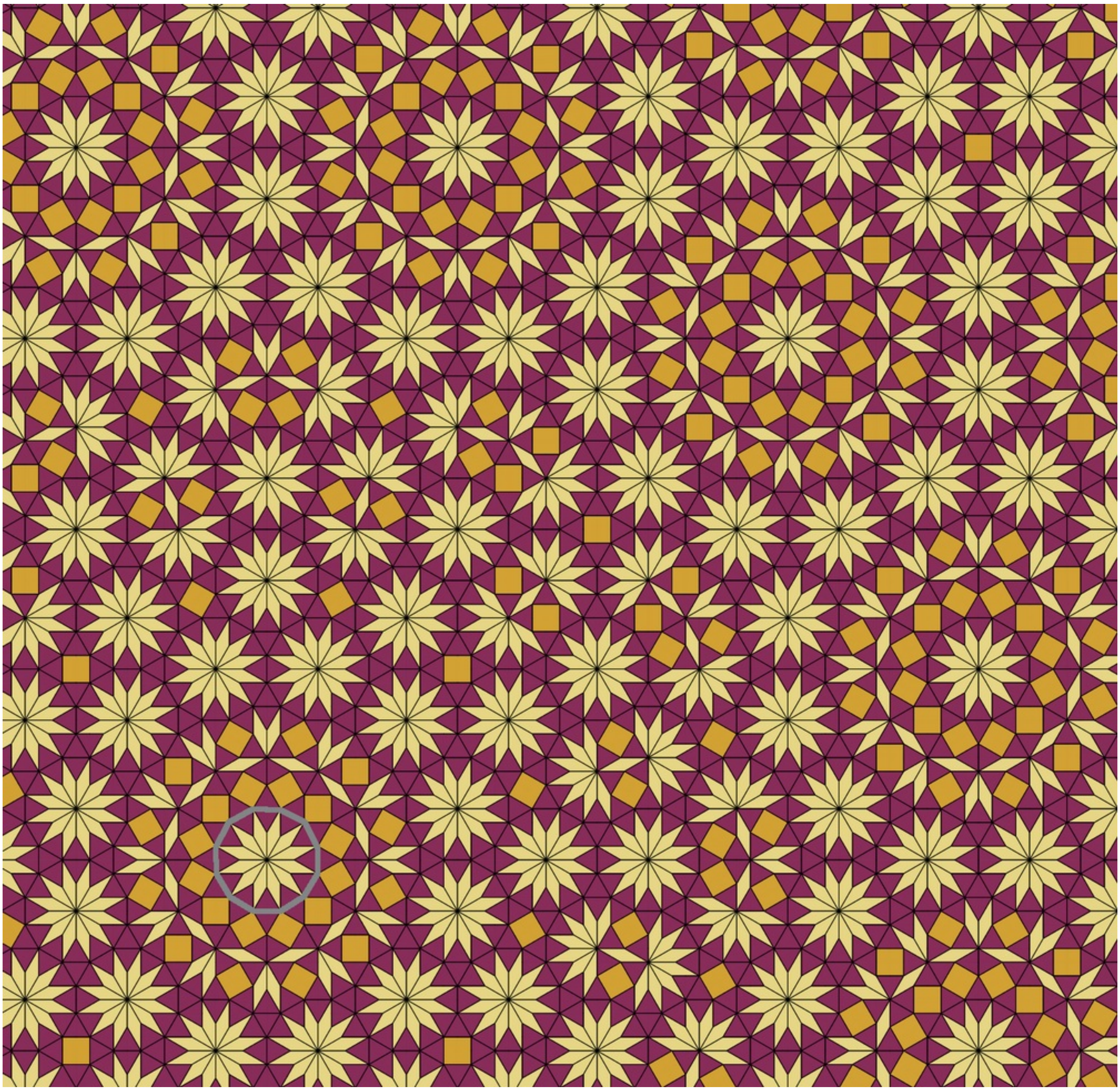
The blue outline is for the 12-fold rosette of the rose window. Note that we already get an extension of this rosette.

We can continue this procedure forever to get a quasiperiodic tiling.

The resulting quasiperiodic tiling

The grey lines show the center of 12-fold rotational symmetry (the initial dodecagon) at the lower left.

Copies of the rosette of the rose window appear "quasiperiodically".



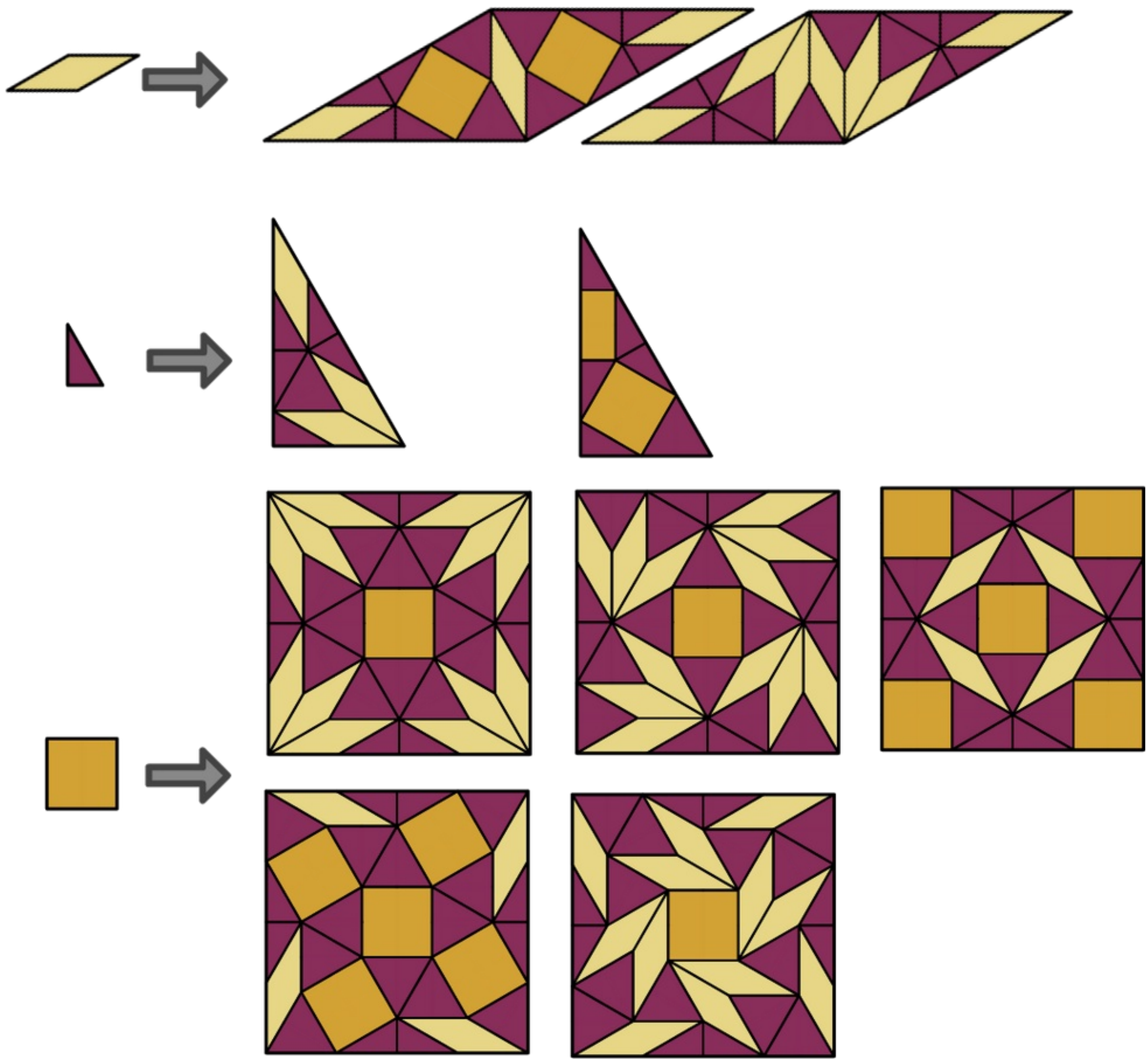
Result of the app at geometricolor.ch/twelvefold2PlusSqrt3/twelveTwoSqrt3.html

Variations

From the 12-fold rosette we can get other substitution rules.

And we can replace two rhombi with squares while moving a triangle.

This gives a lot of different substitution rules resulting in different tilings:



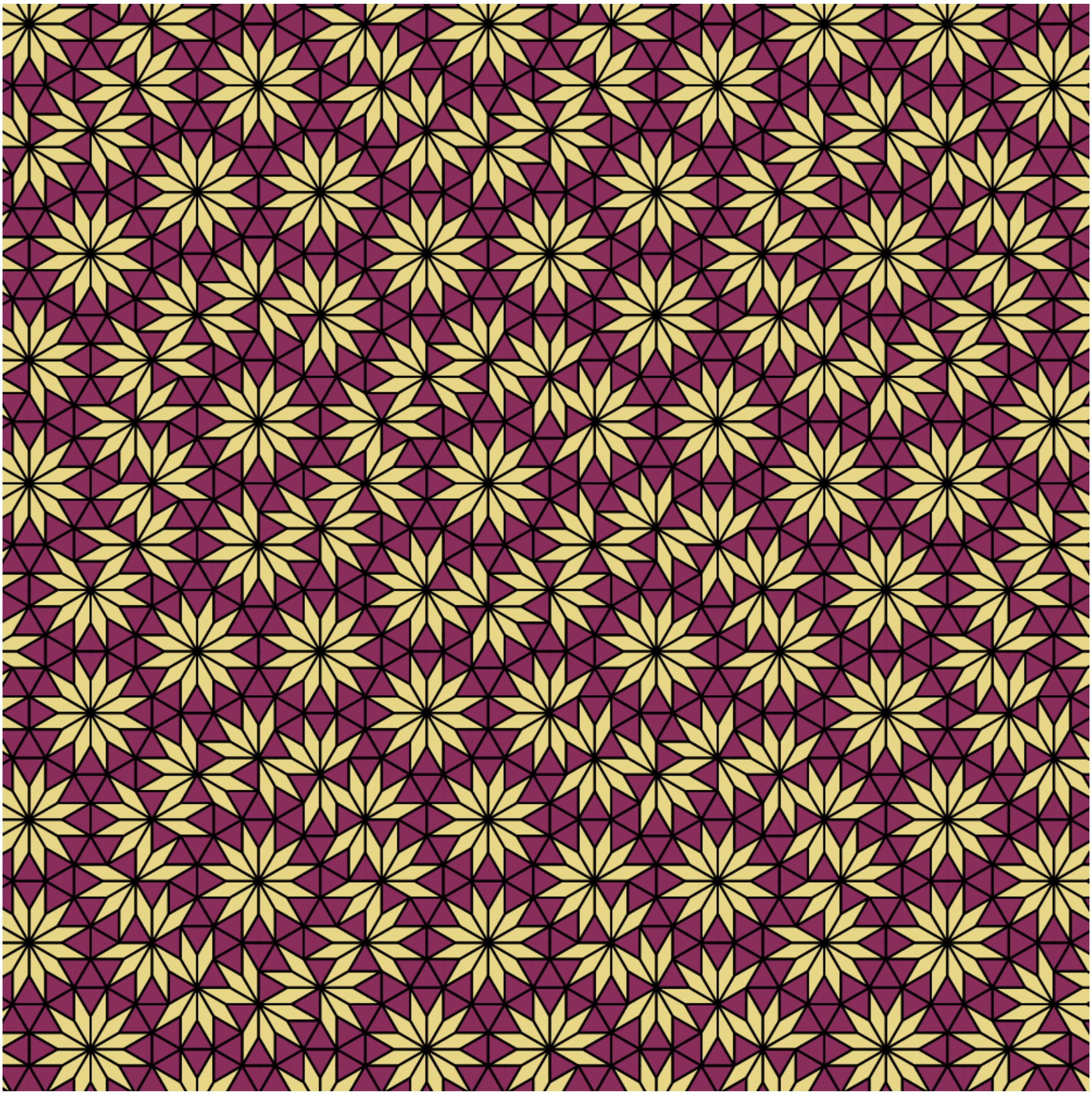
(see also Jim Millar, patternblockhead.com)

Try the app at geometricolor.ch/twelvefold2PlusSqrt3/twelveTwoSqrt3.html

An example

Most choices for the substitutions give quasiperiodic tilings with "less symmetry".

A tiling without squares and without mirror symmetry:



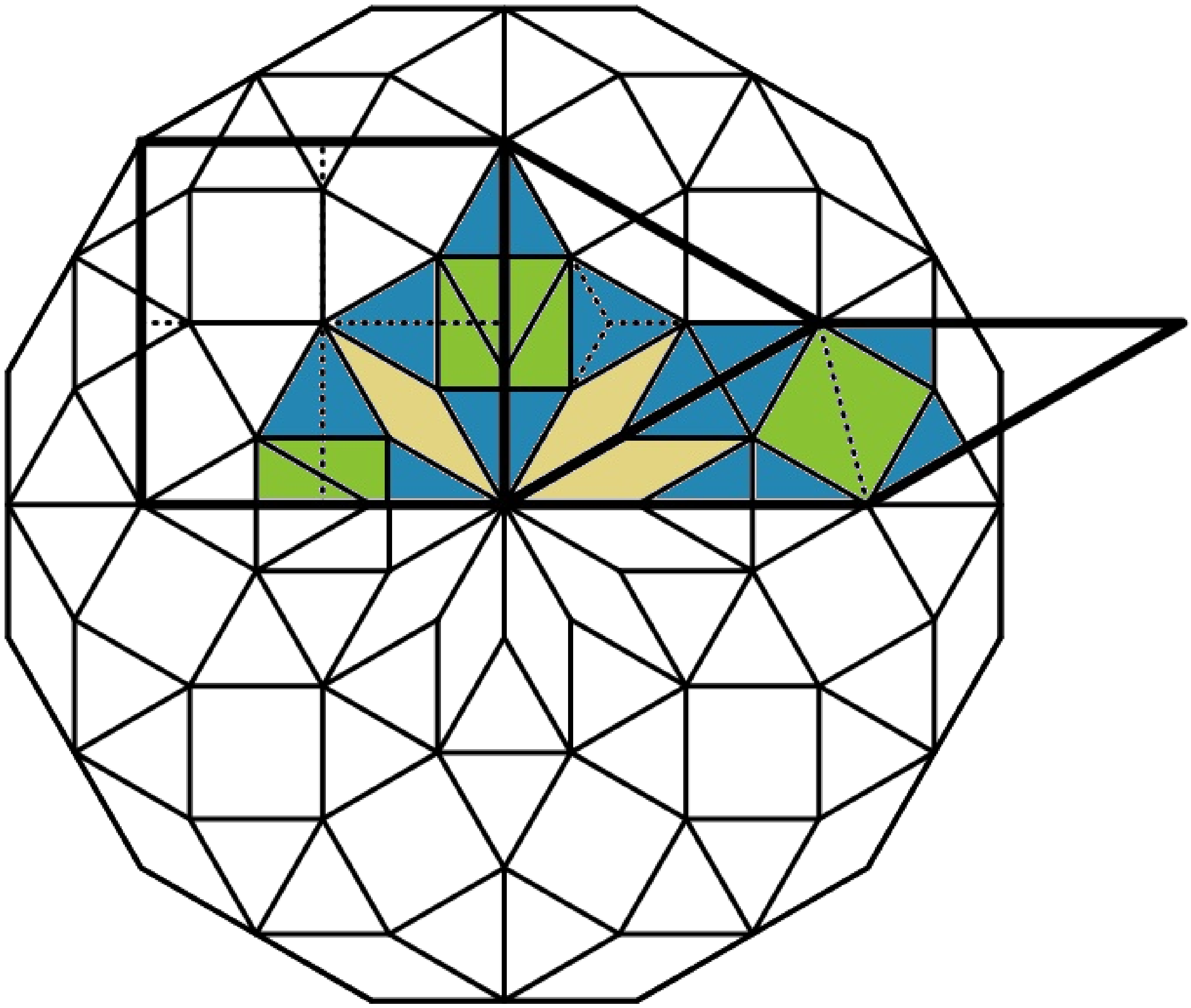
Another inflation factor: 1 + sqrt(3)

Scaled triangles, rhombi and squares fit the rosette rather well for this factor too.

The rosette gives only the substitution rule for the rhombus.

Modifying the rosette we get some ideas for the triangle and the square.

The triangle joins the square and the rhombus.

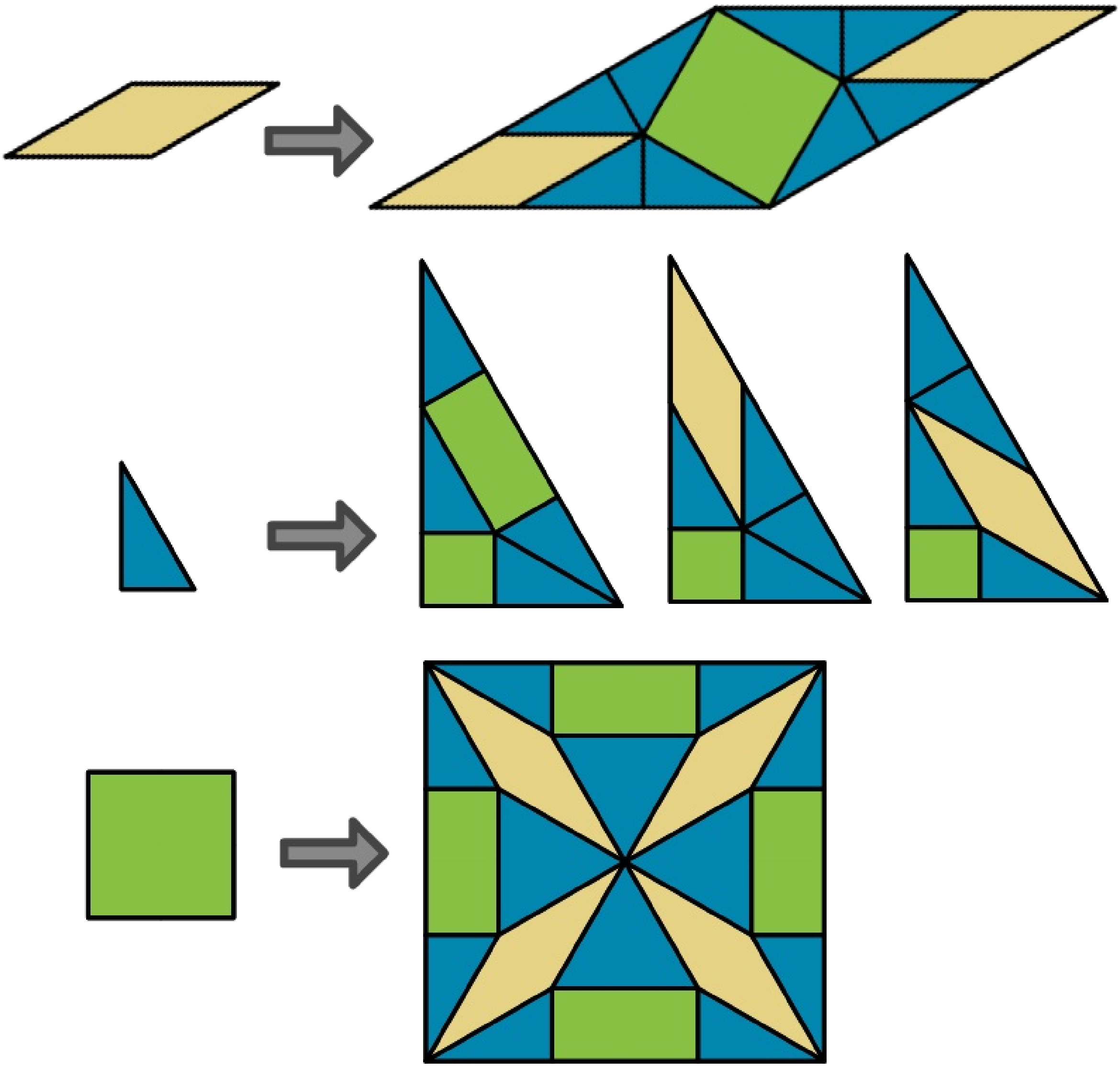


Substitution rules

The substitution rule for the rhombus follows from the rosette.

For the square we can use a four-fold symmetric substitution with mirror symmetry.

The edges of the rhombus and the square do not match. Thus we need three different substitutions for triangles to get tilings with entire squares and equilateral triangles.

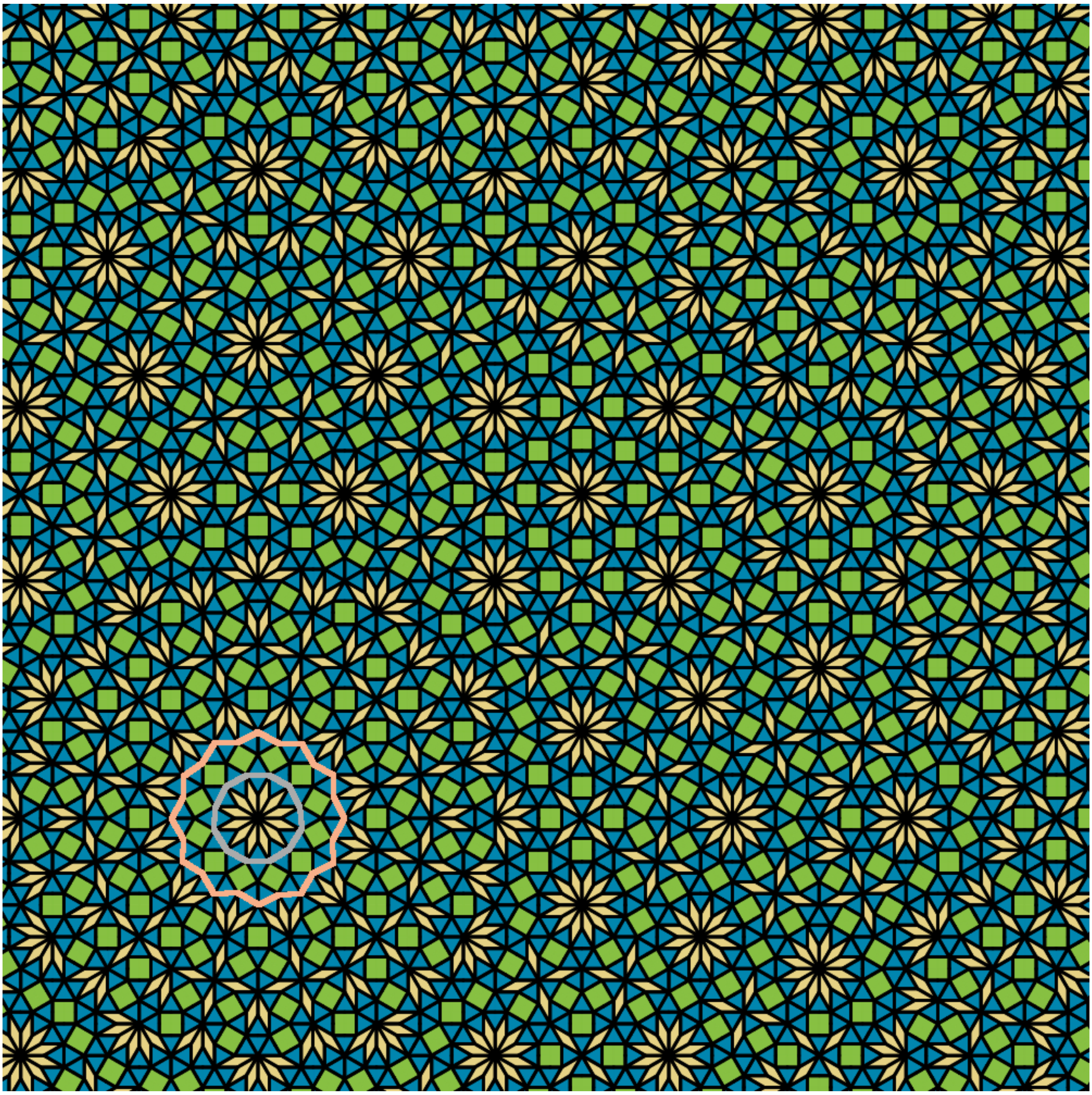


For details see our paper at https://arxiv.org/abs/2102.06046.

The resulting quasiperiodic tiling

The grey lines show the center of 12-fold rotational symmetry (the initial dodecagon) at the lower left.

We get the rosette of the rose window without the outer rhombi (pink lines).
Copies of this fragment appear quasiperiodically.



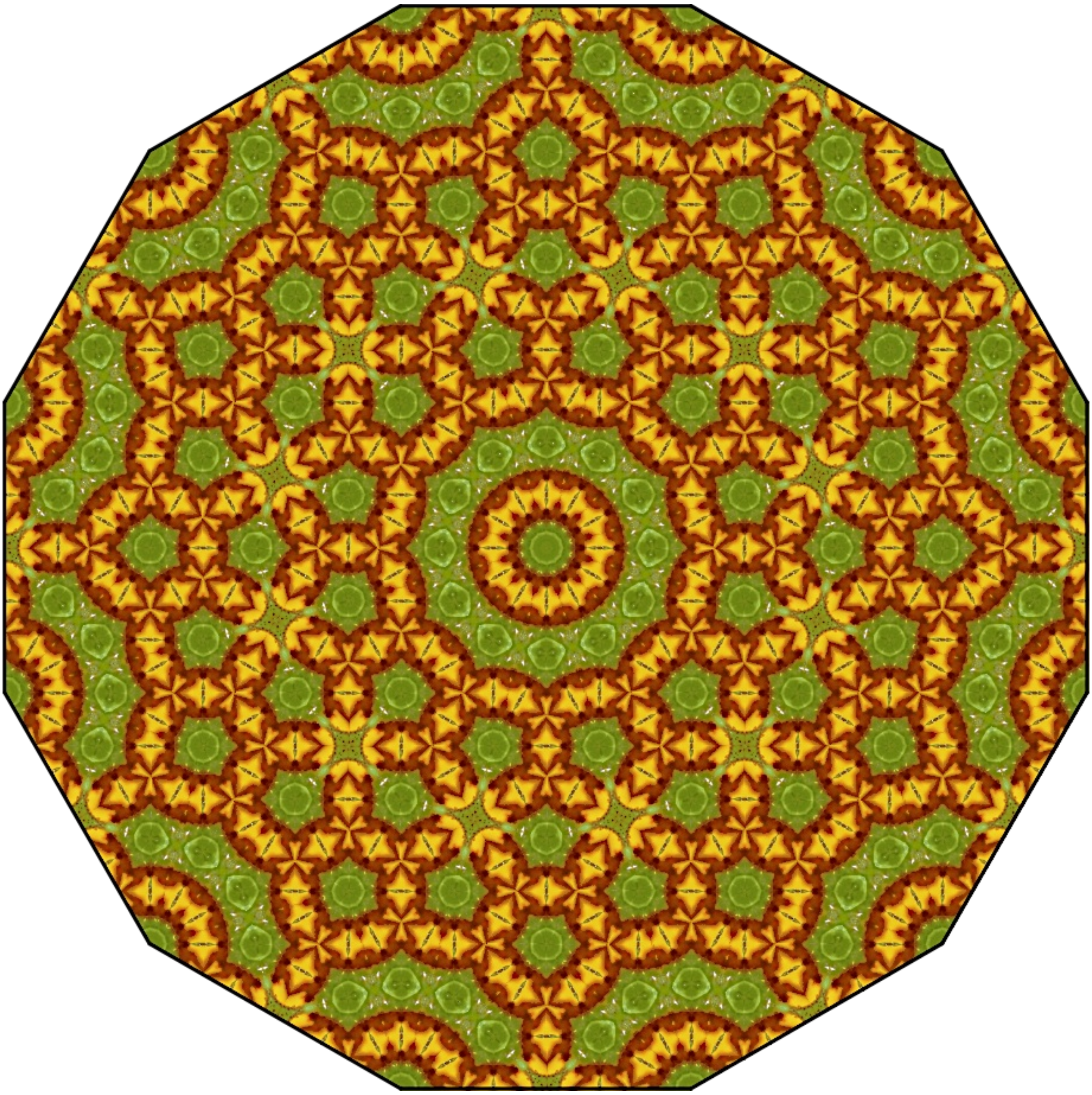
Try the app at geometricolor.ch/twelvefold2PlusSqrt3/twelveOneSqrt3.html

Other substitution rules and tilings are possible.

As a kaleidoscope

We can make kaleidoscopes with the structure of quasiperiodic tilings.

A result with emphasis on the dual lattice:



Generated using geometricolor.ch/twelvefold2PlusSqrt3/twelveOneSqrt3.html

Further results

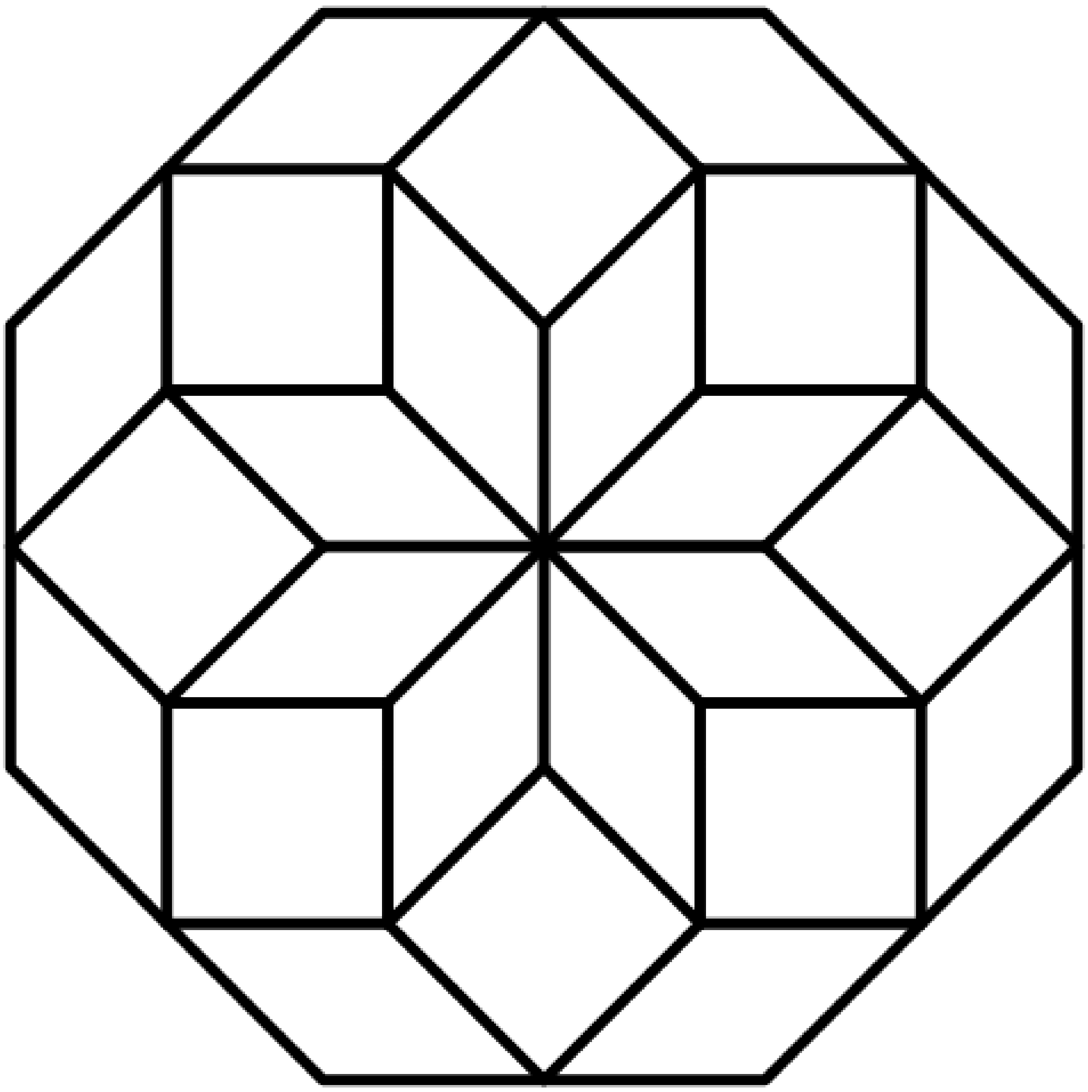
Quasiperiodic tilings with 12-fold rotational symmetry are possible for any inflation ratio of the form n+m*sqrt(3), where n and m are integers larger than zero.

It is important that the square root of 3 is an irrational number. Thus, n and m determine the number of equilateral triangles in the substitution rules.

The equilateral triangles cover half the surface of the tiling for any n and m.

[For details see our paper of this conference.]

One more thing:
Do you see how to get the Ammann-Beenker tiling from this rosette with 8-fold symmetry?



That's all!